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Geometrical symmetry in symmetric Galerkin BEM

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Abstract

We consider a symmetric boundary integral formulation associated with a mixed boundary value problem defined on a domain $\Omega \in \mathbb{R}^2$ with piecewise smooth boundary Γ . We assume that $\bar{\Omega}$ is mapped onto itself by a finite group \mathcal{G} of congruences having at least two distinct elements. Hence, we can decompose the related symmetric Galerkin BEM problem into independent subproblems of reduced dimension with respect to the complete one. Shape functions for each subproblem can be obtained from classical BEM basis, ordered as a vector, applying suitable *restriction matrices* constructed starting from group representation theory.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$, be a bounded domain with a piecewise smooth boundary Γ . The boundary Γ is partitioned into two non intersecting open subset Γ_1 and Γ_2 , with $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \bigcup_{j=1}^J \bar{\Gamma}^j$, Γ^j being an open straight line segments. In the following we always assume $\text{meas } \Gamma_1 > 0$. The solution of the mixed boundary value problem

$$L(x)u(x) = 0 \quad \text{in } \Omega, \quad (1.1)$$

$$u(x) = u^*(x) \quad \text{on } \Gamma_1, \quad q(x) := \frac{\partial u}{\partial \mathbf{n}} = q^*(x) \quad \text{on } \Gamma_2, \quad (1.2)$$

can be expressed by the representation formula

$$u(x) = \int_{\Gamma} U(x, y)q(y) dy - \int_{\Gamma} \frac{\partial}{\partial \mathbf{n}_y} U(x, y)u(y) dy, \quad x \in \Omega. \quad (1.3)$$

In (1.1) $L(\cdot)$ is an elliptic partial differential operator of second order, $U(x, y)$ its fundamental solution (see [4] for a general discussion). In (1.2) $\frac{\partial u}{\partial \mathbf{n}}$ denotes the derivative with respect to the outer normal \mathbf{n} to Γ , and u^* and q^* are given functions. Applications of (1.1)-(1.2) are, for instance, boundary value problems in potential theory and in elastostatic. From (1.3) it is clear that if we want to recover u in Ω we have firstly to know the remaining Cauchy data, since in (1.2) these functions are given only partially. Taking the limit of $u(x)$ for $x \in \Gamma_a$ and the normal derivative $\frac{\partial u}{\partial \mathbf{n}}(x)$ for $x \in \Gamma_2$ in this formula and using the jump relations, one finds the system [2]

$$\int_{\Gamma_1} U(x, y)q(y) dy - \int_{\Gamma_2} \frac{\partial}{\partial \mathbf{n}_y} U(x, y)u(y) dy = f_1(x), \quad x \in \Gamma_1,$$

$$- \int_{\Gamma_1} \frac{\partial}{\partial \mathbf{n}_x} U(x, y) q(y) dy + \int_{\Gamma_2} \frac{\partial^2}{\partial \mathbf{n}_x \partial \mathbf{n}_y} U(x, y) u(y) dy = f_2(x), \quad x \in \Gamma_2. \quad (1.4)$$

In order to perform the Galerkin method, we need a family of finite-dimensional subspaces $\{U_{h,p}(\Gamma)\}$ defined on Γ . Let us define a mesh Γ_h^j for each Γ^j : $\bar{\Gamma}^j = \bigcup_{i=1}^{N_h^j} \bar{\Gamma}_{h,i}^j$ such that $\Gamma_{h,i}^j$ is an open segment. We define for $p \geq 0$, $h > 0$, $U_{h,p}(\Gamma_1)$ to be the set of functions on Γ_1 whose restrictions to $\Gamma^j \subset \Gamma_1$ belong to the set of all polynomials of degree $\leq p$ on $\Gamma_{h,i}^j$. Moreover, for $p \geq 1$, $U_{h,p}^\circ(\Gamma_2)$ will denote those continuous functions on Γ_2 whose restrictions to $\Gamma^j \subset \Gamma_2$ belong to $C^0(\Gamma_2)$ and which vanish at the end points of Γ_2 . The approximating boundary element shape functions of degree $p > 0$ are defined through the standard assembling of the local basis functions defined on each $\Gamma_{h,i}^j$. We then define

$$U_{h,p}(\Gamma) := \text{span} \{(\varphi_i, \psi_\ell) : \varphi_i \in U_{h,p}^\circ(\Gamma_2), \psi_\ell \in U_{h,p}(\Gamma_1)\}. \quad (1.5)$$

The corresponding symmetric Galerkin boundary elements scheme for (1.4) leads to a linear system of the form

$$A\xi = b. \quad (1.6)$$

If the boundary Γ presents symmetry properties, we will exploit them to reduce the computational cost of the solution of (1.6), using a decomposition result for the Galerkin boundary element problem that we will introduce at the end of the next section.

2 Matrix representation of a finite group of congruences and projection operators

Let \mathcal{G} be a finite group of t congruences ($t \geq 2$) of the Euclidean space \mathbb{R}^m ($m = 2, 3$). The group \mathcal{G} can be described by orthogonal matrices γ_i of order m . Let $\{\gamma_1, \dots, \gamma_t\}$ be the elements of \mathcal{G} , γ_1 the identity matrix. From the theory of group representation [5] it follows that any finite group \mathcal{G} admits a finite number q of unitary irreducible, pairwise inequivalent matrix representations

$$\{\omega^{(1)}(\gamma_i)\}, \{\omega^{(2)}(\gamma_i)\}, \dots, \{\omega^{(q)}(\gamma_i)\} \quad (i = 1, \dots, t). \quad (2.1)$$

Let d_ℓ be the order of the representation $\{\omega^{(\ell)}(\gamma_i)\}$, i.e., the order of the matrices $\omega^{(\ell)}(\gamma_i)$. The number q of the representations (2.1) and the orders d_1, \dots, d_q only depend on \mathcal{G} . Any representation $\{\omega^{(\ell)}(\gamma_i)\}$ of order $d_\ell \geq 2$, can be replaced, in the system (2.1), by an equivalent unitary representation. Representations of order 1 are univocally determined. We observe that, if γ_i and γ_j are two elements of \mathcal{G} , then $\omega^{(\ell)}(\gamma_i \gamma_j) = \omega^{(\ell)}(\gamma_i) \omega^{(\ell)}(\gamma_j)$, $\omega^{(\ell)}(\gamma_i^{-1}) = [\omega^{(\ell)}(\gamma_i)]^*$, where $[\omega^{(\ell)}(\gamma_i)]^*$ denote the transpose of the matrix $\omega^{(\ell)}(\gamma_i)$. Always from the theory of group representation it follows that $q \leq t$ and the relation $d_1^2 + d_2^2 + \dots + d_q^2 = t$ holds. Furthermore, $q = t$ if and only if $d_1 = d_2 = \dots = d_q = 1$. Having set $M = d_1 + d_2 + \dots + d_q$, then $q \leq M \leq t$, and we have $q = M = t$ if and only if \mathcal{G} is an abelian group.

Let Ω be a bounded domain in \mathbb{R}^2 with a piecewise smooth boundary Γ , invariant with respect to \mathcal{G} , i.e., sent onto itself by the congruences of \mathcal{G} . Also the boundary Γ is invariant with respect to \mathcal{G} , i.e., for any $\gamma_\ell \in \mathcal{G}$ and $x \in \Gamma$, $(\gamma_\ell x) \in \Gamma$.

Let $\mathcal{W}(\Gamma)$ be the real vector space of real functions defined on Γ . We can associate to any element γ_i of \mathcal{G} a linear transformation T_i defined, for any $v \in \mathcal{W}(\Gamma)$, by

$$(T_i v)(x) := v(\gamma_i^{-1} x) \quad x \in \Gamma, \quad (2.2)$$

where T_i is a linear, invertible transformation from $\mathcal{W}(\Gamma)$ onto $\mathcal{W}(\Gamma)$, and T_1 is the identity.

Definition 2.1 A subset $\mathcal{V}(\Gamma)$ of $\mathcal{W}(\Gamma)$ is said to be invariant with respect to \mathcal{G} (or \mathcal{G} -invariant) if for any $v \in \mathcal{V}(\Gamma)$ and any $\gamma_i \in \mathcal{G}$, $T_i v \in \mathcal{V}(\Gamma)$.

Obviously if v is a function of $\mathcal{W}(\Gamma)$, not identically equal to zero, the set of functions $\{T_i v, i = 1, \dots, t\}$ is invariant with respect to \mathcal{G} .

Definition 2.2 Let \mathcal{L} be a linear operator in $\mathcal{V}(\Gamma)$. We will say that \mathcal{L} is invariant with respect to \mathcal{G} if for any $u \in \mathcal{V}(\Gamma)$: $\mathcal{L} T_i u = T_i \mathcal{L} u$, $i = 1, \dots, t$.

Example 2.3 Let $\mathcal{V}(\Gamma)$ be a suitable Sobolev space and $(\mathcal{L}f)(x) := \int_{\Gamma} \mathcal{K}(x, y) f(y) d\Gamma_y$ an integral operator defined on $\mathcal{V}(\Gamma)$, with kernel $\mathcal{K}(x, y)$.

We have: $T_i(\mathcal{L}f)(x) = \int_{\Gamma} \mathcal{K}(\gamma_i^{-1} x, y) f(y) d\Gamma_y$; since $\gamma_i \in \mathcal{G}$ is an isometry, the mapping $y \rightarrow \gamma_i y$ preserves the differential element $d\Gamma_y$. Thus

$$\mathcal{L}(T_i f)(x) = \int_{\Gamma} \mathcal{K}(x, y) f(\gamma_i^{-1} y) d\Gamma_y = \int_{\Gamma} \mathcal{K}(x, \gamma_i y) f(y) d\Gamma_y.$$

Then the integral operator \mathcal{L} is \mathcal{G} -invariant if the kernel $\mathcal{K}(x, y)$ satisfies the condition $\mathcal{K}(x, y) = \mathcal{K}(\gamma_i x, \gamma_i y)$ for all $x, y \in \Gamma$, $i = 1, \dots, t$.

Starting from the group \mathcal{G} , the system of representation (2.1) and the linear transformations T_i defined by (2.2), we can introduce M linear transformations of $\mathcal{W}(\Gamma)$,

$$P_{\ell k} = \frac{d_{\ell}}{t} \sum_{i=1}^t \omega_{kk}^{(\ell)}(\gamma_i) T_i \quad (\ell = 1, \dots, q; k = 1, \dots, d_{\ell}). \quad (2.3)$$

Owing to the property of the representations (2.1), there holds

$$P_{\ell k}^2 = P_{\ell k}, \quad P_{\ell k} P_{\ell' k'} = 0 \quad \text{if } (\ell, k) \neq (\ell', k'), \quad \sum_{\ell=1}^q \sum_{k=1}^{d_{\ell}} P_{\ell k} = T_1. \quad (2.4)$$

The linear transformations $P_{\ell k}$, which will be called *projection operators*, determine a decomposition of any vector space $\mathcal{V}(\Gamma) \subset \mathcal{W}(\Gamma)$ invariant with respect to \mathcal{G} , into a direct sum of M subspaces $\mathcal{V}_{\ell k}(\Gamma)$; $\mathcal{V}_{\ell k}(\Gamma)$ is the co-domain of $P_{\ell k}$, viewed as a linear transformation from $\mathcal{V}(\Gamma)$ onto itself.

If \mathcal{G} is a non-abelian group, it is useful to consider in the space $\mathcal{W}(\Gamma)$ further linear transformations linked to the system (2.1). Let $\{\omega^{(\ell)}(\gamma_i)\}$ be a representation of \mathcal{G} of order $d_{\ell} \geq 2$. Let us consider d_{ℓ}^2 linear transformations, already introduced in [1], defined as follows

$$A_{kr}^{(\ell)} = \frac{d_{\ell}}{t} \sum_{i=1}^t \omega_{kr}^{(\ell)}(\gamma_i) T_i, \quad k, r = 1, \dots, d_{\ell}. \quad (2.5)$$

If $k = r$, then $A_{kr}^{(\ell)} = P_{\ell k}$.

Definition 2.4 Let $\mathcal{B}(\cdot, \cdot)$ be a bilinear form from $\mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma)$ on \mathbb{R} . We will say that $\mathcal{B}(\cdot, \cdot)$ is \mathcal{G} -invariant if for, any $u, v \in \mathcal{V}(\Gamma)$,

$$\mathcal{B}(T_i u, T_i v) = \mathcal{B}(u, v), \quad i = 1, \dots, t. \quad (2.6)$$

Let $\mathcal{V}(\Gamma)$ be a Hilbert space and let us consider the following problem

$$\text{find } u \in \mathcal{V}(\Gamma) : \mathcal{B}(u, v) = \mathcal{F}(v) \quad \text{for all } v \in \mathcal{V}(\Gamma), \quad (2.7)$$

where $\mathcal{B}(\cdot, \cdot)$ is continuous and coercive, and $\mathcal{F}(\cdot) : \mathcal{V}(\Gamma) \rightarrow \mathbb{R}$ a linear continuous functional. If Γ and $\mathcal{V}(\Gamma)$ are invariant with respect to \mathcal{G} , and $\mathcal{V}(\Gamma) = \bigoplus_{\ell=1}^q \bigoplus_{k=1}^{d_\ell} \mathcal{V}_{\ell k}(\Gamma)$ is the decomposition of $\mathcal{V}(\Gamma)$ defined by the projection operators (2.3) the following fundamental result holds.

Theorem 2.5 If $\mathcal{B}(\cdot, \cdot)$ verifies the condition (2.6) and $P_{\ell k}$ are the projection operators defined in (2.3), then the problem (2.7) can be decomposed into M independent problems; find $u_{\ell k} \in \mathcal{V}_{\ell k}(\Gamma)$ such that

$$\mathcal{B}(u_{\ell k}, v_{\ell k}) = \mathcal{F}(v_{\ell k}) \quad \text{for all } v_{\ell k} \in \mathcal{V}_{\ell k}(\Gamma), \quad \ell = 1, \dots, q; k = 1, \dots, d_\ell. \quad (2.8)$$

The solution of (2.7) can be recovered as $u = \bigoplus_{\ell=1}^q \bigoplus_{k=1}^{d_\ell} u_{\ell k}$.

The above result can be applied, under the invariance hypothesis, in the discrete form to the symmetric Galerkin BEM scheme if we choose the finite dimensional subspace $U_{h,p}(\Gamma)$ defined in (1.5), to be \mathcal{G} -invariant too, and therefore decomposable as $U_{h,p}(\Gamma) = \bigoplus_{\ell=1}^q \bigoplus_{k=1}^{d_\ell} U_{h,p}^{\ell k}(\Gamma)$. Then the symmetric Galerkin boundary element problem can be decomposed into M independent problems which have reduced dimension with respect to the original one and which can be solved on parallel processors. Now one has to construct boundary element basis functions for each subspace $U_{h,p}^{\ell k}(\Gamma)$. With some simple geometries (and groups of congruences) this can be done directly, but in many cases this is a difficult task. We solve it here by applying *restriction matrices*, which we introduce in the next sections, to the basis of $U_{h,p}(\Gamma)$, ordered as a vector. Since there is a one-to-one correspondence between the standard boundary element shape functions and the nodes of the mesh fixed on Γ , in the following we will work directly on the nodes of the boundary.

3 Elementary restriction matrices

In this section we introduce suitable matrices depending only on the group \mathcal{G} and on the system of representations (2.1), which will be called *elementary restriction matrices*. In the following sections we will see how, starting from these, we can construct restriction matrices relative to a mesh defined on Γ . We fix a finite group $\mathcal{G} \equiv \{\gamma_1, \dots, \gamma_t\}$ of congruences of \mathbb{R}^m and a system (2.1) of orthogonal irreducible, pairwise inequivalent representations of \mathcal{G} . \mathcal{G} always admits the representation $\{1, 1, \dots, 1\}$ which we indicate by $\{\omega^{(1)}(\gamma_i)\}$; let us order the remaining representations (2.1) with increasing order d_ℓ ; let $\{\omega^{(1)}(\gamma_i)\}, \dots, \{\omega^{(s)}(\gamma_i)\}$ be the representations of order 1. If \mathcal{G} is an abelian group one has $s = q = t$ and $d_1 = d_2 = \dots = d_t = 1$. If \mathcal{G} is a nonabelian group, it holds $s < q < t$ and therefore $d_1 = d_2 = \dots = d_s = 1$, $2 \leq d_{s+1} \leq \dots \leq d_q$.

Let \mathcal{G} be an abelian group. We will call *elementary restriction matrices* the following t matrices, with 1 row and t columns

$$R_{\ell 1} = \frac{1}{\sqrt{t}} \left(\omega^{(\ell)}(\gamma_1) \cdots \omega^{(\ell)}(\gamma_t) \right), \quad \ell = 1, \dots, t. \quad (3.1)$$

Since representations $\{\omega^{(\ell)}(\gamma_i)\}$ are real, it follows that $\omega^{(\ell)}(\gamma_i) = \pm 1$, for $\ell, i = 1, \dots, t$.

Let \mathcal{G} be a nonabelian group. Correspondingly to the representations $\{\omega^{(\ell)}(\gamma_i)\}$ of order 1 of the system (2.1), we introduce matrices $R_{\ell 1}$ with 1 row and t columns

$$R_{\ell 1} = \frac{1}{\sqrt{t}} \left(\omega^{(\ell)}(\gamma_1) \cdots \omega^{(\ell)}(\gamma_t) \right), \quad \ell = 1, \dots, s. \quad (3.2)$$

We obtain, in this case, s matrices. Let now $\{\omega^{(\ell)}(\gamma_i)\}$ be a representation of the system (2.1) of order d_ℓ , with $d_\ell \geq 2$. With $k = 1, \dots, d_\ell$ fixed, let us consider the following matrix, with d_ℓ rows and t columns

$$R_{\ell k} = \sqrt{\frac{d_\ell}{t}} \begin{pmatrix} \omega_{1k}^{(\ell)}(\gamma_1) & \omega_{1k}^{(\ell)}(\gamma_2) & \cdots & \omega_{1k}^{(\ell)}(\gamma_t) \\ \omega_{2k}^{(\ell)}(\gamma_1) & \omega_{2k}^{(\ell)}(\gamma_2) & \cdots & \omega_{2k}^{(\ell)}(\gamma_t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{d_\ell k}^{(\ell)}(\gamma_1) & \omega_{d_\ell k}^{(\ell)}(\gamma_2) & \cdots & \omega_{d_\ell k}^{(\ell)}(\gamma_t) \end{pmatrix}. \quad (3.3)$$

Due to the orthogonality properties of the representation $\{\omega^{(\ell)}(\gamma_i)\}$, matrix $R_{\ell k}$ has pairwise orthonormal rows. Therefore the rank of matrix $R_{\ell k}$ is d_ℓ . For any representation $\{\omega^{(\ell)}(\gamma_i)\}$ we obtain d_ℓ matrices $R_{\ell k}$ ($k = 1, \dots, d_\ell$). Matrices $R_{\ell k}$ ($\ell = 1, \dots, q$; $k = 1, \dots, d_\ell$) defined in (3.2) and (3.3) will be called *elementary restriction matrices*. The total number of these matrices is M , with $M = d_1 + d_2 + \cdots + d_q$. The matrices defined in (3.1) or (3.2)-(3.3) satisfy some properties, easily deducible from orthogonality relations (2.4) and which we summarise in the following.

Theorem 3.1 ([1]) *The M elementary restriction matrices defined by (3.1) or (3.2)-(3.3) verify the relations*

$$R_{\ell k} R_{\ell k}^* = I_{d_\ell}, \quad R_{\ell k} R_{\ell' k'}^* = 0 \text{ if } (\ell, k) \neq (\ell', k'), \quad \sum_{\ell=1}^q \sum_{k=1}^{d_\ell} R_{\ell k}^* R_{\ell k} = I \quad (3.4)$$

where I_{d_ℓ} , I are identity matrices of order d_ℓ and t respectively.

4 $\mathcal{H}(\Sigma_a)$ spaces and elementary restriction matrices

Let Γ be the piecewise smooth boundary of Ω , invariant with respect to \mathcal{G} , and $a \in \Gamma$. Consider the ordered set

$$\Sigma_a = \{a, \gamma_2^{-1}a, \dots, \gamma_t^{-1}a\}, \quad (4.1)$$

and the space $\mathcal{H}(\Sigma_a)$ of real functions defined in Σ_a . A natural basis B in $\mathcal{H}(\Sigma_a)$ is formed by functions having value 1 in a point of Σ_a and 0 in the remaining points. Having indicated with χ the function of B with value 1 in the point a , we obtain the

ordered basis $B \equiv \{\chi(x), \chi(\gamma_2 x), \dots, \chi(\gamma_t x)\}$, such that, of course,

$$\mathcal{H}(\Sigma_a) = \text{span}\{\chi(x), \chi(\gamma_2 x), \dots, \chi(\gamma_t x)\}. \quad (4.2)$$

$\mathcal{H}(\Sigma_a)$ is a vector space with finite dimension $n \leq t$, invariant with respect to \mathcal{G} (since Σ_a is invariant with respect to \mathcal{G}) and therefore decomposable into direct sum of M subspaces $\mathcal{H}_{\ell k}(\Sigma_a)$. Having set $n_\ell = \dim \mathcal{H}_{\ell k}(\Sigma_a)$, we have $n = \sum_{\ell=1}^q d_\ell n_\ell$.

Definition 4.1 We say that a is a generic point of Γ (with respect to the group \mathcal{G}) if $\dim \mathcal{H}(\Sigma_a) = t$ or, equivalently, if all the elements of Σ_a are distinct.

The following results hold.

Theorem 4.2 ([1]) Having fixed any point $a \in \Gamma$, if $\{\omega^{(\ell)}(\gamma_i)\}$ is a representation of order 1, then $\mathcal{H}_{\ell 1}(\Sigma_a) = \text{span}\{P_{\ell 1}\chi\}$ and $n_\ell \leq 1$. If $\{\omega^{(\ell)}(\gamma_i)\}$ is a representation of order $d_\ell \geq 2$, one has

$$\mathcal{H}_{\ell k}(\Sigma_a) = \text{span}\{A_{k1}^{(\ell)}\chi, \dots, A_{kd_\ell}^{(\ell)}\chi\}, \quad k = 1, \dots, d_\ell, \quad (4.3)$$

and therefore $n_\ell \leq d_\ell$. If a is a generic point, then $n_\ell = d_\ell$ for any ℓ .

Let now V^t be the column vector $(\chi(x), \chi(\gamma_2 x), \dots, \chi(\gamma_t x))^*$, whose order is related to that one fixed for the elements of \mathcal{G} . Corresponding to the representations of order 1 of \mathcal{G} , for the elementary restriction matrices defined in (3.1), (3.2) we have $R_{\ell 1}V^t = \sqrt{t}P_{\ell 1}\chi$. From Theorem 4.2, it follows that

$$\mathcal{H}_{\ell 1}(\Sigma_a) = \text{span}\{R_{\ell 1}V^t\}. \quad (4.4)$$

Corresponding to the representations of order $d_\ell \geq 2$, for the elementary restriction matrices defined in (3.3) we have $R_{\ell k}V^t = \sqrt{t/d_\ell} \left(A_{k1}^{(\ell)}\chi, A_{k2}^{(\ell)}\chi, \dots, A_{kd_\ell}^{(\ell)}\chi \right)^*$. From (4.3), it follows that

$$\mathcal{H}_{\ell k}(\Sigma_a) = \text{span}\{R_{\ell k}V^t\}. \quad (4.5)$$

In both cases, if a is a generic point, the components of the vector $R_{\ell k}V^t$ constitute a basis in $\mathcal{H}_{\ell k}(\Sigma_a)$. Therefore, for any generic point a , the elementary restriction matrix $R_{\ell k}$ represents the projection operator $P_{\ell k}$ from $\mathcal{H}(\Sigma_a)$ onto $\mathcal{H}_{\ell k}(\Sigma_a)$, if we choose V^t as a basis in $\mathcal{H}(\Sigma_a)$.

Now, we want to construct elementary restriction matrices $R_{\ell k}$ which represent the projection operators $P_{\ell k}$ from $\mathcal{H}(\Sigma_a)$ onto $\mathcal{H}_{\ell k}(\Sigma_a)$ for nongeneric points. Therefore let us suppose a to be a nongeneric point, i.e., such that the functions

$$\chi(x), \chi(\gamma_2 x), \dots, \chi(\gamma_t x) \quad (4.6)$$

are linearly dependent. Let n be the maximum number of linearly independent functions among (4.6) and let the following functions be linearly independent,

$$\chi(\gamma_{i_1} x), \dots, \chi(\gamma_{i_n} x). \quad (4.7)$$

It is convenient to order the functions (4.7) with increasing index i_α ; therefore let us suppose $i_1 < i_2 < \dots < i_n$. In this case elementary restriction matrices $R_{\ell k}$ will have n columns. The number n_ℓ of rows ($n_\ell \leq d_\ell$) of each $R_{\ell k}$ is not determined by i_1, i_2, \dots, i_n .

In general, we only can say that matrices $R_{\ell 1}, \dots, R_{\ell d_\ell}$ have the same number n_ℓ of rows, where $n_\ell = \dim \mathcal{H}_{\ell k}(\Sigma_a)$.

Then we now consider a significant class of nongeneric points. Having fixed ℓ ($\ell = 2, \dots, t$), let $I_\ell(\Gamma)$ be the set of all points $a \in \Gamma$ such that

$$a = \gamma_\ell^{-1} a. \quad (4.8)$$

From (4.8) it follows, for any $i : \chi(\gamma_i x) = \chi(\gamma_\ell \gamma_i x)$. This implies that the functions (4.6) are naturally subdivided into subsets and any subset contains coincident functions. Then we can obtain elementary restriction matrices for the space $\mathcal{H}(\Sigma_a)$ with $a \in I_\ell(\Gamma)$ starting from elementary restriction matrices built in Section 3, with the following procedure,

- Let us sum to each column of index i_α ($\alpha = 1, \dots, n$) all the columns of index j , with j such that $\gamma_j^{-1} a = \gamma_{i_\alpha}^{-1} a$. We indicate with $\tilde{R}_{\ell k}$ the obtained matrices, all with d_ℓ rows and n columns, but not all full-rank matrices; some of these may be zero matrices.
- Let us extract from *nonzero* matrices $\tilde{R}_{\ell k}$ submatrices $\bar{R}_{\ell k}$ made up of n_ℓ linearly independent rows.
- Finally, let us construct from $\bar{R}_{\ell k}$ matrices $R_{\ell k}$ with a row-orthonormalization procedure.

The (nonzero) matrices $R_{\ell k}$ verify the properties expressed by Theorem 3.1. Furthermore, matrices $R_{\ell k}$, applied to the vector $V^n = (\chi(\gamma_{i_1} x), \dots, \chi(\gamma_{i_n} x))^*$ corresponding to a point $a \in I_\ell(\Gamma)$, give vectors whose components constitute a basis for $\mathcal{H}_{\ell k}(\Sigma_a)$. For this reason they represent the projection operators from $\mathcal{H}(\Sigma_a)$ onto $\mathcal{H}_{\ell k}(\Sigma_a)$, for any $a \in I_\ell(\Gamma)$. Then we will say that the matrices $R_{\ell k}$, with n_ℓ rows and n columns, are *elementary restriction matrices* for the space $\mathcal{H}(\Sigma_a)$ relative to points $a \in I_\ell(\Gamma)$. Furthermore $n = \sum_{\ell=1}^q d_\ell n_\ell$.

5 $\mathcal{H}(\Sigma)$ spaces and restriction matrices

Let Γ be the piecewise smooth boundary of Ω , Σ a set formed by N points of Γ constituting a not necessarily uniform mesh defined on Γ . Let us suppose Γ and Σ invariant with respect to \mathcal{G} . Let $\mathcal{H}(\Sigma)$ be the vector space of real functions defined in Σ . $\mathcal{H}(\Sigma)$ is a N -dimensional vector space, invariant with respect to \mathcal{G} ; this is due to the fact that Σ is invariant with respect to \mathcal{G} . A natural basis B in $\mathcal{H}(\Sigma)$, invariant with respect to \mathcal{G} , is formed by functions having value 1 in a point of Σ and 0 in the remaining points. In order to more easily construct restriction matrices for the space $\mathcal{H}(\Sigma)$, or equivalently for the mesh Σ , it is suitable to introduce in the set Σ the following equivalence relation.

Definition 5.1 *We say that a point a' is equivalent to a'' if there exists an element $\gamma_i \in \mathcal{G}$ such that $a'' \equiv \gamma_i^{-1} a'$ (and therefore $a' \equiv \gamma_i a''$).*

The points of the set Σ are then subdivided into r equivalence classes. If $r = 1$ one has $\mathcal{H}(\Sigma) = \mathcal{H}(\Sigma_a)$, with $a \in \Sigma$. Then let us suppose $r \geq 2$. We order the points of the set Σ as follows; having indicated with a_1, \dots, a_r r pairwise inequivalent points of Σ , we consider the following ordered points

$$a_1, \gamma_2^{-1} a_1, \dots, \gamma_t^{-1} a_1, a_2, \gamma_2^{-1} a_2, \dots, \gamma_t^{-1} a_2, \dots, a_r, \gamma_2^{-1} a_r, \dots, \gamma_t^{-1} a_r. \quad (5.1)$$

If points (5.1) are distinct, we have $N = rt$. If some points among (5.1) coincide, we will erase from the sequence (5.1) a point if it is equal to a previous one. Then a sequence of N points, with $N < rt$, will remain, with $n^{(1)}$ points equivalent to a_1 , $n^{(2)}$ equivalent to $a_2, \dots, n^{(r)}$ equivalent to a_r . In both cases $\mathcal{H}(\Sigma) = \mathcal{H}(\Sigma_{a_1}) \oplus \mathcal{H}(\Sigma_{a_2}) \oplus \dots \oplus \mathcal{H}(\Sigma_{a_r})$, with $\dim \mathcal{H}(\Sigma_{a_j}) = n^{(j)} \leq t$, $j = 1, \dots, r$ and $N = n^{(1)} + n^{(2)} + \dots + n^{(r)}$. We indicate by $C_{\ell k}^{(j)}$ the elementary restriction matrices relative to the space $\mathcal{H}(\Sigma_{a_j})$, constructed as indicated in Section 4. Let $n_\ell^{(j)}$ be the number of rows of the matrix $C_{\ell k}^{(j)}$; having fixed j , the number of columns of matrices $C_{\ell k}^{(j)}$, for any ℓ and k , is $n^{(j)}$. We consider therefore the following M block matrices

$$\tilde{R}_{\ell k} = \begin{pmatrix} C_{\ell k}^{(1)} & O & O & \dots & O \\ O & C_{\ell k}^{(2)} & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & C_{\ell k}^{(r)} \end{pmatrix}, \quad (5.2)$$

with $\tilde{N}_\ell = n_\ell^{(1)} + n_\ell^{(2)} + \dots + n_\ell^{(r)}$ rows and N columns, from which we have to eliminate the possible zero rows. Matrices $R_{\ell k}$ determined by this procedure, which we call *restriction matrices* for the space $\mathcal{H}(\Sigma)$ of dimension N , have rank equal to the number N_ℓ of the remained rows and for these matrices properties expressed in Theorem 3.1 still hold. In both cases, we have the following theorem.

Theorem 5.2 *Considering the basis B in $\mathcal{H}(\Sigma)$ as a column vector V^N with the order deduced from (5.1), the components of the vector $R_{\ell k} V^N$ form a basis in $\mathcal{H}_{\ell k}(\Sigma)$. Therefore the M matrices $R_{\ell k}$, having fixed in $\mathcal{H}(\Sigma)$ the ordered basis V^N , determine a decomposition of $\mathcal{H}(\Sigma)$ in M subspaces, which coincides with the one obtained with the projection operators $P_{\ell k}$.*

Preliminary numerical results appear promising; algorithms for potential and linear elasticity problems are being implemented on parallel processors to analyse the efficiency of the proposed approach.

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